Priors comparison in Bayesian mediation framework with binary outcome

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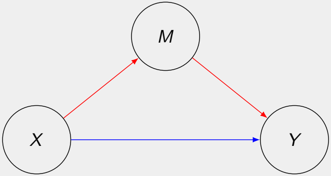
Introduction

Mediation

- outcome variable Y.
- exposure variable X.
- mediator variable M.



- The purpose is to decompose the causal effect of X on Y into two parts
  - The effect that passes through the mediator M ($\rightsquigarrow$ indirect effect)
  - The effect that does not ($\rightsquigarrow$ direct effect).

Motivation of my works

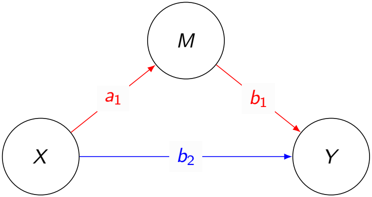
- Practical issues in psychology.
- Propose alternative statistical solutions: $\rightsquigarrow$ Bayesian framework

Gaussian model: case of correlation

- The most widely model to evaluate the effect of X on Y is the linear regression:

$$Y = \psi_0 + \psi X + \varepsilon, \quad \varepsilon \sim \mathcal{N}(0, \sigma^2)$$

- ψ measures the association between X and Y.



In presence of the mediator M,

- the linear model is of the form:

$$\begin{aligned} Y &= b_0 + b_1 M + b_2 X + \varepsilon_2 \\ M &= a_0 + a_1 X + \varepsilon_1 \\ &\quad \text{where } \varepsilon_i \sim \mathcal{N}(0, \sigma_i^2),\ i = 1, 2. \end{aligned}$$

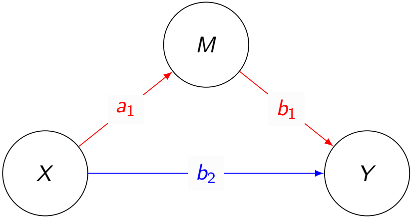
$$\Longrightarrow \quad Y = b_0 + b_1 a_0 + (a_1 b_1 + b_2) X + a_1 \varepsilon_1 + \varepsilon_2$$

- The total effect is

$$\psi = \underbrace{b_2}_{\text{Direct}} + \underbrace{a_1 b_1}_{\text{Indirect}} .$$

Mediation with binary outcome

- outcome variable $Y \in \{0, 1\}$.
- exposure variable X.
- mediator variable $M \in \mathcal{M} \subset \mathbb{R}$.



$$\begin{cases} \mathbb{E}(Y|X, M) = \dfrac{1}{1 + e^{-(b_0 + b_1 M + b_2 X)}} \\ \\ M = a_0 + a_1 X + \varepsilon, \qquad \varepsilon \sim \mathcal{N}(0, \sigma^2 I) \end{cases}$$

Problem: Decomposition of the effect of X on Y in the logistic case.

Remark

The decomposition of the total effect is not valid for logistic mediation model.

- ψ measures the total effet

$$\mathbb{E}(Y|X) = \frac{1}{1 + e^{-(\psi_0 + \psi X)}} \qquad (*)$$

- b_2 measures the direct effect

$$\mathbb{E}(Y|X, M) = \frac{1}{1 + e^{-(b_0 + b_1 M + b_2 X)}} \qquad (**)$$

But

$$\psi \neq b_2 + a_1 b_1$$

If we replace M by $M = a_0 + a_1 X + \varepsilon$ in (**), we do not obtain (*)

Causal Effect

- We want to define the causal effect of the exposure $X \in \{0,1\}$ on the outcome Y.
- Y_x is the potential outcome if $X = x, \quad x \in \{0,1\}$.
- The average causal effect of X on Y is

$$ACE = \mathbb{E}(Y_1 - Y_0)$$

Theorem

Assume that

1. consistency assumption

$$\text{if } X = x, \text{ then } Y_x = Y,$$

2. ignorability assumption,

$$Y_x \perp\!\!\!\perp X \text{ for } x \in \{0,1\},$$

We have

$$ACE = \mathbb{E}(Y|X=1) - \mathbb{E}(Y|X=0)$$

In presence of a mediator Pearl (2001)

1. Direct effect is the difference between the potential outcomes Y_1 and Y_0 but imagining that the mediator is blocked at its value M_x
    1. Pure Natural Direct Effect when $x = 0$,
    $$NDE(0) = \mathbb{E}(Y_{1,M_0} - Y_{0,M_0})$$
    2. Total Natural Direct Effect, when $x = 1$,
    $$NDE(1) = \mathbb{E}(Y_{1,M_1} - Y_{0,M_1})$$
2. Indirect effect is the difference between the potential outcome Y_x but imagining that the mediator corresponds to M_1 and to M_0
    1. Pure Natural Indirect Effect when $x = 0$,
    $$NIE(0) = \mathbb{E}(Y_{0,M_1} - Y_{0,M_0})$$
    2. Total Natural Indirect Effect, when $x = 1$,
    $$NIE(1) = \mathbb{E}(Y_{1,M_1} - Y_{1,M_0})$$

A simple calculation provides

$$ACE = PNDE + TNIE = PNIE + TNDE$$

Identifiability of NDE and NIE

- The potential outcome $Y_{x,M_{x^*}}$ is never observed since $x \neq x^*$.
- Additional assumptions are required to obtain the identifiability of the effects (see for instance Pearl, 2001; Imai et al., 2010).
- Under these, we have, for $x, x^* \in \{0, 1\}$,

$$\mathbb{E}(Y_{x,,M_{x^*}}) = \int_{-\infty}^{+\infty} \mathbb{E}(Y|X = x, M = m) f_{M|X=x^*}(m) dm$$

$\rightsquigarrow$

$$NDE(x) = \int \left[\mathbb{E}(Y|X = 1, M = m) - \mathbb{E}(Y|X = 0, M = m)\right] \mathrm{d}\mathbb{P}_{M|X=x}(m)$$

$$NIE(x) = \int \mathbb{E}(Y|X = x, M = m)\mathrm{d}\mathbb{P}_{M|X=1}(m) - \int \mathbb{E}(Y|X = x, M = m)\mathrm{d}\mathbb{P}_{M|X=0}(m)$$

Gaussian example:

$NDE(0) = NDE(1) = b_2,\ NIE(0) = NIE(1) = a_1 b_1.$

Bayesian inference of effects NDE(x) et NIE(x)

$$\begin{cases} M = a_0 + a_1 X + \varepsilon & \alpha = (a_0, a_1) \\ \mathbb{E}(Y|X, M) = \dfrac{1}{1 + e^{-(b_0 + b_1 M + b_2 X)}} & \beta = (b_0, b_1, b_2) \end{cases}$$

- Likelihood function :
  $f(Y, M|\alpha, \beta, \sigma^2, X) = \Phi_1(Y|\beta, M, X)\Phi_2(M|\alpha, \sigma^2, X)$

$$\text{where } \begin{cases} \Phi_1(Y|\beta, M, X) = \dfrac{\exp(Y(b_0 + b_1 M + b_2 X))}{1 + \exp(b_0 + b_1 M + B_2 X)} \\ \Phi_2 \text{ Gaussian } \mathcal{N}(a_0 + a_1 X, \sigma^2 I_n) \end{cases}$$

- prior distribution of α, β, σ^2.
- parameters of Interest:

$$\text{NDE}_\theta(x) = \frac{1}{\sigma\sqrt{2\pi}} \int \left[\frac{1}{1 + e^{-(\beta_0 + \beta_1 m + \beta_2)}} - \frac{1}{1 + e^{-(\beta_0 + \beta_1 m)}} \right] e^{-\frac{1}{2\sigma^2}(m - \alpha_0 - \alpha_1 x)^2} \mathrm{d}m$$

$$\text{NIE}_\theta(x) = \frac{1}{\sigma\sqrt{2\pi}} \int \frac{1}{1 + e^{-(\beta_0 + \beta_1 m + \beta_2 x)}} \left[e^{-\frac{1}{2\sigma^2}(m - \alpha_0 - \alpha_1)^2} - e^{-\frac{1}{2\sigma^2}(m - \alpha_0)^2} \right] \mathrm{d}m.$$

$\rightsquigarrow$ Non-explicit function of θ

Construction of the prior

We propose strategies with different degrees of information :

- a weakly G-prior introduced by Zellner (1971).
- an informative prior resulting from Launay et al. (2015) transfer Learning

		linear regression		Logistic regression
		σ^2	α	β
G-prior	weakly info.	$\frac{1}{\sigma^2}\mathbb{1}_{\mathbb{R}^+}$	$G-$prior	
Informative	info.		Launay et al. (2015)	

G-priors

- For linear regression model (Zellner, 1971)

$$Y = \mathbf{X}\beta + \varepsilon, \quad \varepsilon \sim \mathcal{N}(0, \sigma^2 I_n), \mathbf{X} = [\mathbb{1}, X_1, ..., X_p].$$

  G-prior is define as follows

$$\begin{cases} \beta|\sigma^2, \mathbf{X} \sim \mathcal{N}_{p+1}\left(\widetilde{\beta}, g\sigma^2(\mathbf{X}'\mathbf{X})^{-1}\right) \\ \pi(\sigma^2|\mathbf{X}) \propto \sigma^{-2} \end{cases}$$

  ⇝ The choice $\widetilde{\beta} = 0$ and $g = n$ gives to prior information the same weight as an observation.

- Generalisation to logistic regression (Marin and Robert, 2007)

$$\beta|\mathbf{X} \sim \mathcal{N}_{p+1}\left(\widetilde{\beta}, g(\mathbf{X}'W\mathbf{X})^{-1}\right), \quad W = diag(p_i(1-p_i)).$$

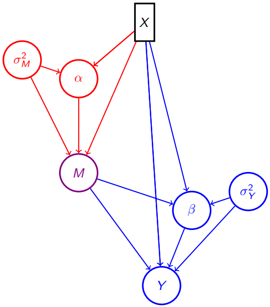
  ⇝ The choice $\widetilde{\beta} = 0$ and $p_i = 1/2$ gives a weak information to the prior.

G-priors and mediation model

- Decomposition of the joint distribution :

$$f(M, Y, \alpha, \sigma^2, \beta|X) = \Phi_1(Y|\beta, \sigma_Y^2, X, M)\pi(\beta, \sigma_Y^2|X, M)$$
$$\Phi_2(M|\alpha, \sigma_M^2, X)\pi(\alpha, \sigma^2|X).$$

- $\pi(\beta|X, M)$ $G-$prior logistic regression $\mathbb{E}(Y|X, M) = \frac{1}{1+e^{-(b_0+b_1M+b_2X)}}$
- $\pi(\alpha, \sigma^2|X)$ $G-$prior linear regression $M = a_0 + a_1X + \varepsilon_M$.



Remark on the Gaussian mediation model

- Galharret and Philippe (2021) use the same strategy for the Gaussian mediation model.
- Nuijten et al. (2015) address this problem as two independent Gaussian regression models.

Testing direct and indirect effects

Proposition

In the mediation model we have:

$$\text{NDE}(0) = 0 \iff \text{NDE}(1) = 0 \iff b_2 = 0$$

$$\text{NIE}(0) = 0 \iff \text{NIE}(1) = 0 \iff a_1 b_1 = 0$$

Test for direct effect ($b_2 = 0$) the likelihood ratio test

Test for indirect effect ($a_1 b_1 = 0$) bootstrap is used to approximate confidence interval for $a_1 b_1 \rightsquigarrow$ the performances are not very good for small samples.

Alternative

Test for the indirect effect :

$$\mathcal{H}_0 : \mathrm{NIE}_\theta(x) = 0 \text{ against } \mathcal{H}_1 : \mathrm{NIE}_\theta(x) \neq 0.$$

Decision rule : let $\mathcal{I}_\alpha$ be the credible interval of $\mathrm{NIE}_\theta(x)$

$$P(NIE_\theta(x) \in \mathcal{I}_\alpha | Y, M) = 1 - \alpha$$

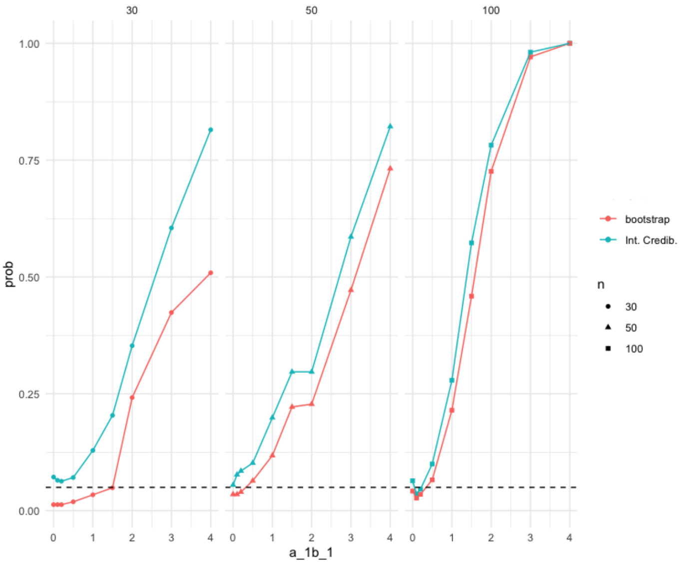
$$0 \notin \mathcal{I}_\alpha \rightsquigarrow \text{The absence of indirect effect is rejected}$$

Remark

- *According to Berstein von Mises theorem, for all $\theta \in \Theta$ such as $a_1 b_1 = 0$, the frequentist probability satisfies*

$$P_\theta(0 \in \mathcal{I}_\alpha) \to \alpha$$

Numerical results for indirect effect



$a_1 b_1 = 0$ gives the empirical level

Informative model M_1 (Launay et al., 2015)

We assume that

1. A historical data are available $\mathcal{D}_h$ and
2. A Bayesian analysis has been done on $\mathcal{D}_h$

$$
\begin{aligned}
m_h =& \mathbb{E}\left(\alpha_h, \beta_h | \mathcal{D}_h\right) \\
\Sigma_h =& \mathbb{V}ar\left(\alpha_h, \beta_h | \mathcal{D}_h\right)
\end{aligned}
$$

Application: longitudinal study: historical data come from the previous time step.
Assumption: Only small changes in parameters between the two studies.
Prior distribution of (α, β) **:**

$$(\alpha, \beta) \sim \mathcal{N}_5(K m_h, g\Sigma_h)$$

where $K = \begin{pmatrix} k_1 & & \\ & \ddots & \\ & & k_J \end{pmatrix}$ et $k_1, ..., k_J$ are i.i.d. $k_j \sim \mathcal{N}(1, \tau^2)$.

$\rightsquigarrow$ g et τ^2 hyperparameters of the model.

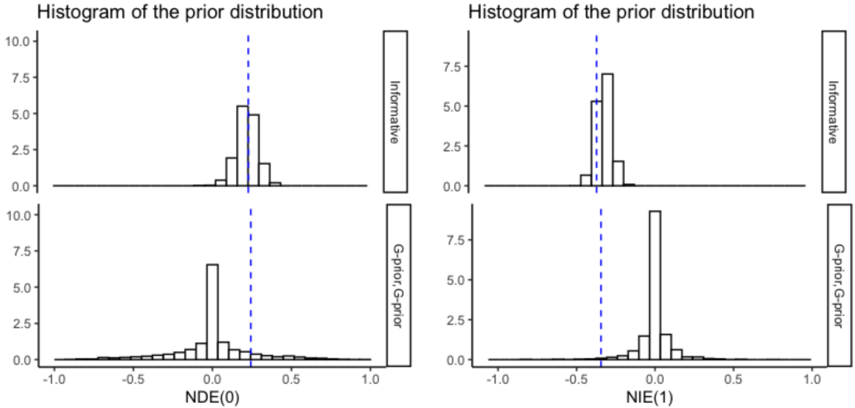
Illustration

Historical data : $n_h = 100$, $\alpha_h = (1, -2)$ et $\beta_h = (-0.5, 1.5, 1)$.

data : We simulate samples with the following parameters

	α	β	NDE(0)	NIE(1)
Case 1	$(1, -2)$	$(-0.5, 1, 1.5)$	0.24	-0.35
Case 2	$(1, -1)$	$(-1, 2, 0.5)$	0.07	-0.29
Case 3	$(1, 1)$	$(1, 1, -1)$	-0.15	0.15

Comparison of priors on NDE(0) and NIE(1). case 1



Numerical results

		NDE(0)=PNDE				NIE(1)=TNIE			
N	model	bias	RMSE	coverage	length	bias	RMSE	coverage	length
CASE 1: Time-invariant parameters									
30	*info*	0.01	0.11	0.97	0.48	0.01	0.11	0.96	0.43
30	*G-prior*	-0.02	0.16	0.96	0.63	0.04	0.12	0.95	0.48
50	*info*	0.00	0.09	0.96	0.40	0.01	0.08	0.96	0.34
50	*G-prior*	-0.01	0.12	0.95	0.49	0.02	0.09	0.96	0.37
100	*info*	0.00	0.07	0.96	0.30	0.01	0.06	0.96	0.24
100	*G-prior*	0.00	0.07	0.96	0.30	0.01	0.06	0.96	0.24
CASE 2: Time-varying parameters with invariance of the sign of the effects									
30	*info*	0.03	0.09	0.97	0.39	0.02	0.10	0.96	0.40
30	*G-prior*	0.00	0.13	0.93	0.50	0.02	0.11	0.96	0.43
50	*info*	0.02	0.07	0.97	0.32	0.01	0.08	0.96	0.33
50	*G-prior*	0.00	0.10	0.94	0.40	0.02	0.08	0.96	0.34
100	*info*	0.00	0.06	0.98	0.24	0.01	0.06	0.94	0.24
100	*G-prior*	-0.01	0.07	0.96	0.28	0.01	0.06	0.95	0.24
CASE 3: Time-varying parameters with changing sign of the effects									
30	*info*	0.04	0.16	0.94	0.59	-0.04	0.10	0.92	0.36
30	*G-prior*	0.00	0.16	0.95	0.60	-0.01	0.10	0.95	0.39
50	*info*	0.03	0.13	0.94	0.48	-0.03	0.08	0.94	0.29
50	*G-prior*	0.00	0.14	0.94	0.49	-0.01	0.08	0.95	0.3
100	*info*	0.00	0.07	0.96	0.30	0.01	0.06	0.96	0.24
100	*G-prior*	0.00	0.07	0.96	0.30	0.01	0.06	0.96	0.24

Application in psychology

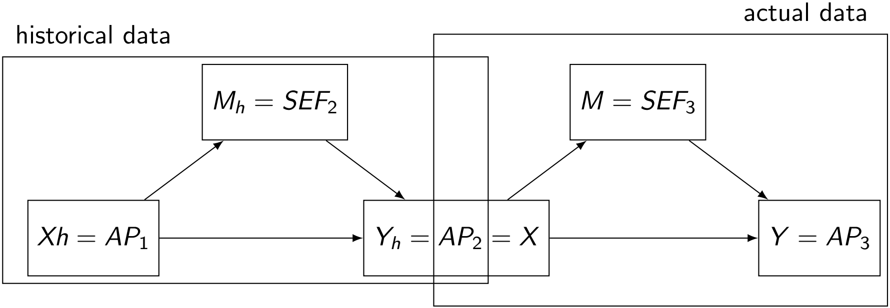
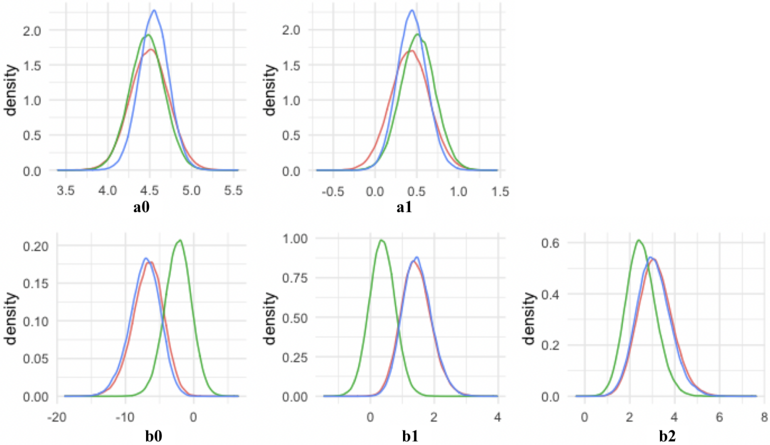


Figure: Structural model. *AP*: Academic Performance, *SEF*: Self Efficacy Feeling.

Estimation of the regression coefficients



Estimation of the effects

	G-prior			*info*		
	Estimate	Lower	Upper	Estimate	Lower	Upper
PNIE	0.094	-0.014	0.211	0.114	0.016	0.222
TNIE	0.056	-0.011	0.140	0.058	0.003	0.128
PNDE	0.473	0.277	0.664	0.497	0.296	0.696
TNDE	0.434	0.245	0.634	0.441	0.241	0.646

Table: 95 %-Credible Interval for the effects at the second time measurement

Conclusion

1. Bayesian estimation of the direct and indirect effects:
   the posterior distribution of both effects can be calculated
2. Procedure to include information coming from historical study
   Improvement of the precision and the accuracy
3. Testing procedure for the effects
   Improvement of the significance level and the power comparing to bootstrap approximation

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